

Limit of a Function

ISI & CMI Yearlong Course 2024

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① Introduction to Limits

- ① Limit Points (already done in Topology)
a.k.a. cluster points.

We need limit points for defining limits!

② Definitions & Basic Properties

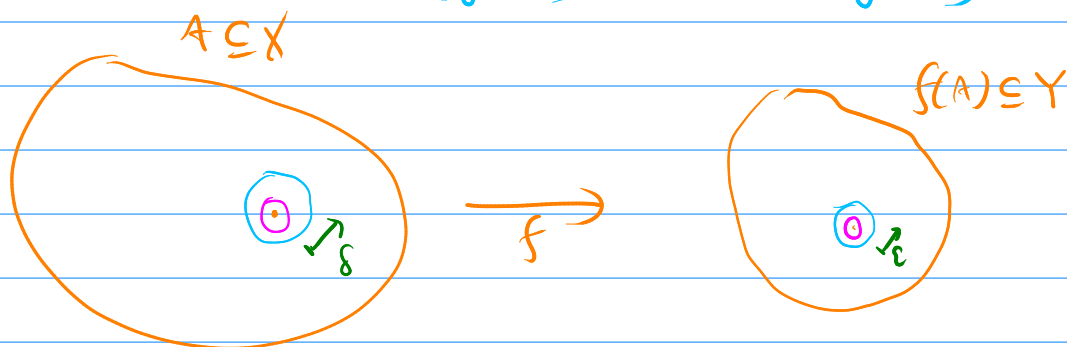
- Let X, Y be metric space, $f: A \subseteq X \rightarrow Y$.
Let c be a limit point of A .

possibly depending on ε

$$\exists L \in Y \mid \forall \varepsilon > 0, \exists \delta > 0 \mid$$
$$d_X(x, c) < \delta \Rightarrow d_Y(f(x), L) < \varepsilon$$

This is written as: $f(x) \rightarrow L$ as $x \rightarrow c$ or
 $\lim_{x \rightarrow c} f(x) = L$

... and spoken as: " $f(x)$ approaches L as x approaches c "
(goes to) (goes to)



We say $f(x)$ has limit L @ $x=c$.

- eg: $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$

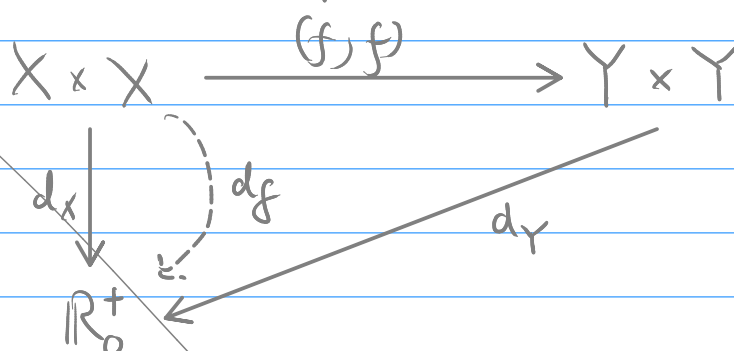
- Sequential Definition of Limits.

↗ agrees with the sequential notion of limit pts.

$$\lim_{x \rightarrow c} f(x) = L \iff \forall x_n \rightarrow c; \forall n, x_n \neq c \quad f(x_n) \rightarrow L$$

(Try the proof yourself, present in Bartle-Shorhott)

X, Y metric spaces, $f: X \rightarrow Y$



In some sense, we are locally checking for the "commutativity" of the above diagram ...

If d_X, d_Y are induced by a norm, they are bilinear forms, hence, under some conditions, the diagram is commutative locally. i.e. the metrics are equivalent up to isomorphism.

(Continuity!)

• Algebra of Limits Given everything exists---

$$i) \lim_{x \rightarrow c} f(x) + \lambda g(x) = \lim_{x \rightarrow c} f(x) + \lambda \lim_{x \rightarrow c} g(x)$$

$$ii) \lim_{x \rightarrow c} f(x)g(x) = \left(\lim_{x \rightarrow c} f(x) \right) \left(\lim_{x \rightarrow c} g(x) \right)$$

$$iii) \lim_{x \rightarrow c} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow c} f(x)}{\lim_{x \rightarrow c} g(x)} \quad \text{if } g \text{ non-vanishing in a nbd of } c$$

• Theorem:

$$f, g: A \subseteq X \rightarrow \mathbb{R} \quad | \quad f(x) \leq g(x) \quad \forall x \in A$$

$$\text{Then:} \quad \lim_{x \rightarrow c} f(x) \leq \lim_{x \rightarrow c} g(x) \quad \text{given both exist.}$$

Corollary: (Sandwich Theorem)

⊙ Standard Limits

(read from any JEE style book)

⊙ Divergence Criteria

i) Unbounded f

ii) 2 sequences converge to different values.

iii) Infinite oscillations! $\hookrightarrow \sin\left(\frac{1}{x}\right)$ @ 0.

No limit exists.

⊛ Extensions to the Concept of Limits (in $\mathbb{R}$)

⊙ Infinite Limits

- We say $f(x) \rightarrow \infty$ as $x \rightarrow c$ iff
 $\forall M > 0, \exists \delta > 0 \mid |x - c| < \delta \Rightarrow f(x) > M$

- (similarly for $-\infty$)

⊙ Limits @ $\pm\infty$

- We say $f(x) \rightarrow L$ as $x \rightarrow \infty$ iff
 $\forall \varepsilon > 0, \exists M > 0 \mid x > M \Rightarrow |f(x) - L| < \varepsilon$

- (similarly for $-\infty$)

⊙ One-Sided Limits

- We say $\lim_{x \rightarrow c^-} f(x) = L$ (left hand limit)

$$\iff \forall x_n \rightarrow c \mid \forall n \ x_n < c, f(x_n) \rightarrow L$$

- (similarly for right-hand limit)

- We say $f(x) \uparrow L$ or $f(x) \rightarrow L^-$ iff
 $f(x_n) \rightarrow L$ and $\forall n \ f(x_n) < L$.
usually, monotone.

- (similarly for $f(x) \downarrow L$ or $f(x) \rightarrow L^+$)